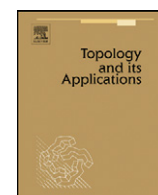




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## On certain properties of the spaces of order-preserving functionals

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## ABSTRACT

In the present paper it is proved that the functor  $O_\tau$  of  $\tau$ -smooth order preserving functionals and the functor  $O_R$  of Radon order preserving functionals, do not change the weight of infinite Tychonoff spaces. It is shown that the density and the weak density of infinite Tychonoff spaces do not increase under these functors. Moreover, if  $X$  is a metric space with the second axiom of countability then the spaces  $O_\tau(X)$  and  $O_R(X)$  are also metrizable.

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## 0. Introduction

Let  $X$  be a compact ( $\equiv$  compact Hausdorff topological space) and let  $C(X)$  be the Banach algebra of all continuous real-valued functions with the usual algebraic operations and with the sup-norm. For functions  $\varphi, \psi \in C(X)$  we shall write  $\varphi \leq \psi$  if  $\varphi(x) \leq \psi(x)$  for all  $x \in X$ . If  $c \in \mathbb{R}$  then by  $c_X$  we denote the constant function identically equal to  $c$ . We recall that a functional  $\mu : C(X) \rightarrow \mathbb{R}$  is said [2] to be:

- (1) *order-preserving* if for any pair  $\varphi, \psi \in C(X)$  of functions the inequality  $\varphi \leq \psi$  implies  $\mu(\varphi) \leq \mu(\psi)$ ;
- (2) *weakly additive* if  $\mu(\varphi + c_X) = \mu(\varphi) + c\mu(1_X)$  for all  $\varphi \in C(X)$  and  $c \in \mathbb{R}$ ;
- (3) *normed* if  $\mu(1_X) = 1$ .

For a compact  $X$  we denote by  $W(X)$  the set of all order-preserving weakly additive functionals  $\mu : C(X) \rightarrow \mathbb{R}$ , and by  $O(X)$  we denote the set of all normed functionals from  $W(X)$ . Note that according to Proposition 1 in [7] each order-

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preserving weakly additive functional is continuous. Further order-preserving weakly additive functionals are called order-preserving functionals [2].

Let  $X$  be a Tychonoff space and let  $C_b(X)$  be the algebra of all bounded continuous real-valued functions with the pointwise algebraic operations. For a function  $\varphi \in C_b(X)$  put  $\|\varphi\| = \sup\{|\varphi(x)| : x \in X\}$ .  $C_b(X)$  with this norm is a Banach algebra. For a net  $\{\varphi_\alpha\} \subset C_b(X)$   $\varphi_\alpha \downarrow 0_X$  means that for every point  $x \in X$  one has  $\varphi_\alpha(x) \geq \varphi_\beta(x)$  at  $\beta > \alpha$  and  $\lim_\alpha \varphi_\alpha(x) = 0_X$ . In this case we say that  $\{\varphi_\alpha\}$  is a monotone decreasing net pointwise convergent to zero.

For a Tychonoff space  $X$  we denote by  $\beta X$  its Stone–Čech compact extension. Given any function  $\varphi \in C_b(X)$  consider its continuous extension  $\tilde{\varphi} \in C(\beta X)$ . This gives an isomorphism between the spaces  $C_b(X)$  and  $C(\beta X)$ . Moreover,  $\|\tilde{\varphi}\| = \|\varphi\|$ , i.e. this isomorphism is an isometry, and the topological properties of the above spaces coincide. Therefore one may consider any function from  $C_b(X)$  as an element of  $C(\beta X)$ . Hence Definitions 0.1 and 0.2 from [1] may be given in the following form.

**Definition 1.** An order-preserving functional  $\mu \in W(\beta X)$  is said to be  $\tau$ -smooth if  $\mu(\varphi_\alpha) \rightarrow 0$  for each monotone net  $\{\varphi_\alpha\} \subset C(\beta X)$  decreasing to zero on  $X$ .

**Definition 2.** An order-preserving functional  $\mu \in O(\beta X)$  is said to be Radon order-preserving functional if  $\mu(\varphi_\alpha) \rightarrow 0$  for each bounded net  $\{\varphi_\alpha\} \subset C(\beta X)$  which uniformly converges to zero on compact subsets of  $X$ .

For a Tychonoff space  $X$  we shall denote by  $W_\tau(X)$  resp.  $W_R(X)$  the sets of all  $\tau$ -smooth resp. Radon order-preserving functionals from  $W(\beta X)$ . The sets  $W_\tau(X)$  and  $W_R(X)$  are equipped with the pointwise convergence topology. The base of neighborhoods of a functional  $\mu \in W_\tau(X)$  (respectively, of  $\mu \in W_R(X)$ ) in the pointwise convergence topology consists of the sets

$$\langle \mu; \varphi_1, \dots, \varphi_k; \varepsilon \rangle = \{v \in W(\beta X) : |v(\varphi_i) - \mu(\varphi_i)| < \varepsilon\} \cap W_\tau(X)$$

$$(\text{respectively, } \langle \mu; \varphi_1, \dots, \varphi_k; \varepsilon \rangle = \{v \in W(\beta X) : |v(\varphi_i) - \mu(\varphi_i)| < \varepsilon\} \cap W_R(X)),$$

where  $\varphi_i \in C(\beta X)$ ,  $i = 1, \dots, k$ , and  $\varepsilon > 0$ .

Put

$$O_\tau(X) = \{\mu \in W_\tau(X) : \mu(1_X) = 1\},$$

$$O_R(X) = \{\mu \in W_R(X) : \mu(1_X) = 1\}.$$

The operations  $O_\tau$  and  $O_R$  are functors [1, Theorem 0.3] in the category *Tych* of Tychonoff spaces and their continuous maps.

Let  $A$  be a closed subset of the compact  $X$ . Since the functor  $O : \text{Comp} \rightarrow \text{Comp}$  is monomorphic [2], i.e. preserves embeddings of closed subsets of compacta, we may assume that  $O(A)$  is a subset of the compact  $O(X)$ . Moreover the embedding of  $O(A)$  into  $O(X)$  is given by means of the mapping  $O(i_A) : O(A) \rightarrow O(X)$ , where  $i_A : A \rightarrow X$  is the identical embedding. Therefore the following notion is well defined.

Given a closed subset  $A$  of the compact  $X$  an order-preserving functional  $\mu \in O(X)$  is said to be *supported* on  $A$  if  $\mu \in O(A)$  [2]. The set

$$\text{supp } \mu = \bigcap \{A : \mu \in O(A) \text{ and } A \text{ is closed in } X\}$$

is called the *support* of the order-preserving functional  $\mu$ .

For a Tychonoff space  $X$  put

$$O_\beta(X) = \{\mu \in O(\beta X) : \text{supp } \mu \subset X\}.$$

The operation  $O_\beta$  translating a Tychonoff space  $X$  to  $O_\beta(X)$ , is a functor [3] in the category *Tych*. Obviously the inclusions

$$O_\beta(X) \subset O_R(X) \subset O_\tau(X) \subset O(\beta X) \tag{1}$$

are valid for any Tychonoff space  $X$ , and the equalities

$$O_\beta(X) = O_R(X) = O_\tau(X) = O(\beta X)$$

are true for arbitrary compact  $X$ .

Let  $X$  and  $Y$  be compacta and let  $f : X \rightarrow Y$  be a continuous map. Then the map  $O(f) : O(X) \rightarrow O(Y)$  defined by the formula

$$O(f)(\mu)(\varphi) = \mu(\varphi \circ f)$$

is continuous, where  $\varphi \in C(Y)$  and  $\mu \in O(X)$ .

Now let  $X$  and  $Y$  be Tychonoff spaces and let  $f : X \rightarrow Y$  be a continuous map. Put

$$O_\tau(f) = O(\beta f)|_{O_\tau(X)}, \quad O_R(f) = O(\beta f)|_{O_R(X)}$$

and

$$O_\beta(f) = O(\beta f)|_{O_\beta(X)},$$

where  $\beta f : \beta X \rightarrow \beta Y$  is the Stone–Čech extension of  $f$ .

Note that the above maps  $O_\tau(f) : O_\tau(X) \rightarrow O_\tau(Y)$ ,  $O_R(f) : O_R(X) \rightarrow O_R(Y)$  and  $O_\beta(f) : O_\beta(X) \rightarrow O_\beta(Y)$  are defined correctly and they also are continuous.

The following statement has been established in [2].

**Proposition 0.**

- (1) For every compact  $X$  the space  $O(X)$  of order-preserving functionals is also compact.
- (2) The functor  $O$  of order-preserving functionals, acting in the category of compact spaces and their continuous mappings, preserves the weight of an infinite compact.

In the paper [3] it was shown that the functor  $O_\beta$  also preserves the weight of infinite Tychonoff spaces. There exists an example [4, Example 3], which shows that under the functor  $O(\beta \cdot)$  the weight of a Tychonoff space may strictly increase, more precisely, the example shows that the functor translates a Tychonoff space  $X$  with countable weight to a compact  $O(\beta X)$  with continuum weight. The question whether the functors  $O_\tau$  and  $O_R$  preserve the weight remained open. In this paper we obtain a positive answer to this question. Moreover we prove that under the functors  $O_\tau$  and  $O_R$  the density and the weak density of infinite Tychonoff spaces do not increase. It also is established that the weak densities of the following spaces coincide:

- $O_\omega(X)$  of order-preserving functionals with finite supports,
- $O_\beta(X)$  of order-preserving functionals with compact supports,
- $O_R(X)$  of Radon order-preserving functionals,
- $O_\tau(X)$  of  $\tau$ -smooth order-preserving functionals and
- $O(\beta X)$  of all order-preserving functionals.

Note that the space  $W_\tau(X)$  equipped with the pointwise convergence topology may be considered as a subspace of the topological product  $\Pi = \prod \{\mathbb{R}_\varphi : \varphi \in C(\beta X)\}$  of real lines  $\mathbb{R}_\varphi = \mathbb{R}$ . Since  $\Pi$  is a Tychonoff space the spaces  $W_\tau(X)$  and  $O_\tau(X)$  with the pointwise convergence topology are also Tychonoff spaces.

Further, in [9] it was proved that if a space  $X$  is metrizable or completely metrizable then so is positive cone  $M_\tau^+(X)$  of the spaces of all  $\tau$ -smooth linear functionals  $\mu : C_b(X) \rightarrow \mathbb{R}$ .

Therefore, the following question naturally arises.

**Question.** Given a metrizable space  $X$ , is the space  $O_\tau(X)$  of  $\tau$ -smooth order-preserving functionals also metrizable?

We give a positive answer to this question in a particular case. More precisely, it is established that for a metric space  $X$  with the second axiom of countability the space  $O_\tau(X)$  of  $\tau$ -smooth order-preserving functionals, and the space  $O_R(X)$  of Radon order-preserving functionals are also metrizable. It should be noted that the approach of [9] in the proof of metrizability of the positive cone  $M_\tau^+(X)$  of the spaces of all  $\tau$ -smooth linear functionals on  $C_b(X)$  cannot be applied in this case.

## 1. Main results

Let  $X$  be a topological space. Recall that the *weight* of  $X$  is the cardinal number  $w(X)$  defined by the formula

$$w(X) = \min\{|\mathfrak{B}| : \mathfrak{B} \text{ is a base of the topology on } X\}.$$

In this section we shall prove that the functors  $O_\tau$  of  $\tau$ -smooth order-preserving functionals and  $O_R$  of Radon order-preserving functionals preserve the weight of infinite Tychonoff spaces. To do this, we need some constructions.

Let  $Y$  be a subspace of a Tychonoff space  $X$ . Put

$$C = \{\psi|Y : \psi \in C_b(X)\}.$$

The following notion is well known. A subspace  $Y \subset X$  is called *C\*-embedded* in  $X$  if for each function  $\varphi \in C_b(Y)$  there exists a function  $\tilde{\varphi} \in C_b(X)$  such, that  $\tilde{\varphi}|Y = \varphi$ . If  $Y$  is a C\*-embedded subspace of the given space  $X$  then clearly  $C \equiv C_b(Y)$ .

For a C\*-embedded subspace  $Y$  of a Tychonoff space  $X$ , a functional  $\mu \in W(\beta X)$  and a function  $\varphi \in C_b(Y)$  put

$$r_Y^X(\mu)(\varphi) = \inf\{\mu(\psi) : \psi \in C_b(X), \psi \geq (\inf\{\varphi(y) : y \in Y\})_X, \psi|Y = \varphi\}. \quad (2)$$

**Lemma 1.** For each  $\mu \in W(\beta X)$  we have  $r_Y^X(\mu) \in W(\beta Y)$ . In other words  $r_Y^X(\mu)$  is an order-preserving weakly additive functional on  $C_b(Y)$ .

**Proof.** We have the following equality

$$r_Y^X(\mu)(1_Y) = \mu(1_X), \quad (3)$$

which directly follows from (2). Let  $\varphi \in C_b(Y)$ . Then  $\varphi + c_Y \in C$  for all  $c \in \mathbb{R}$ . We have

$$\begin{aligned} r_Y^X(\mu)(\varphi + c_Y) &= \inf\{\mu(\psi): \psi \in C_b(X), \psi \geq (\inf\{\varphi(y) + c: y \in Y\})_X, \psi|_Y = \varphi + c_Y\} \\ &= \inf\{\mu(\psi): \psi \in C_b(X), \psi \geq (\inf\{\varphi(y): y \in Y\})_X + c_X, \psi|_Y = \varphi + c_Y\} \\ &= \inf\{\mu(\psi - c_X) + c \cdot \mu(1_X): \psi \in C_b(X), \psi - c_X \geq (\inf\{\varphi(y): y \in Y\})_X, (\psi - c_X)|_Y = \varphi\} \\ &= \inf\{\mu(\psi - c_X): \psi \in C_b(X), \psi - c_X \geq (\inf\{\varphi(y): y \in Y\})_X, (\psi - c_X)|_Y = \varphi\} + c \cdot \mu(1_X) \\ &= r_Y^X(\mu)(\varphi) + c \cdot \mu(1_X) \\ &= (\text{by virtue of (3)}) \\ &= r_Y^X(\mu)(\varphi) + c \cdot r_Y^X(\mu)(1_Y), \end{aligned}$$

i.e.  $r_Y^X(\mu)(\varphi + c_Y) = r_Y^X(\mu)(\varphi) + c \cdot r_Y^X(\mu)(1_Y)$ .

Now let us show that  $r_Y^X(\mu)$  is an order-preserving functional. Let  $\varphi_i \in C_b(Y)$ ,  $i = 1, 2$ , and  $\varphi_1 \leq \varphi_2$ . Then

$$\begin{aligned} r_Y^X(\mu)(\varphi_1) &= \inf\{\mu(\psi): \psi \in C_b(X), \psi \geq (\inf\{\varphi_1(y): y \in Y\})_X, \psi|_Y = \varphi_1\} \\ &\leq \inf\{\mu(\psi): \psi \in C_b(X), \psi \geq (\inf\{\varphi_2(y): y \in Y\})_X, \psi|_Y = \varphi_1\} \\ &\leq \inf\{\mu(\psi): \psi \in C_b(X), \psi \geq (\inf\{\varphi_2(y): y \in Y\})_X, \psi|_Y = \varphi_2\} \\ &= r_Y^X(\mu)(\varphi_2), \end{aligned}$$

i.e.  $r_Y^X(\mu)(\varphi_1) \leq r_Y^X(\mu)(\varphi_2)$ . Thus the functional  $r_Y^X(\mu)$  is order-preserving and weakly additive on  $C_b(Y)$ . Lemma 1 is proved.  $\square$

The order-preserving functional  $r_Y^X(\mu)$  defined as above is said to be a *restriction* of the given order-preserving functional  $\mu \in W(\beta X)$  and the map  $r_Y^X: W(\beta X) \rightarrow W(\beta Y)$  is called the *restriction operator*.

From Lemma 1 and the equality (3) we have immediately the following

**Proposition 1.** Let  $Y$  be a  $C^*$ -embedded subspace of Tychonoff space  $X$ . Then  $r_Y^X(\mu) \in O(\beta Y)$  if and only if  $\mu \in O(\beta X)$ .

Let  $Y \subset X$ ,  $\mu \in W(\beta Y)$  and  $\varphi \in C_b(X)$ . Put

$$e_X^Y(\mu)(\varphi) = \mu(\varphi|_Y). \quad (4)$$

The following statement is obvious.

**Proposition 2.** For every  $\mu \in W(\beta Y)$  we have  $e_X^Y(\mu) \in W(\beta X)$  and  $e_X^Y(\mu)(1_X) = \mu(1_Y)$ . Hence,  $e_X^Y(\mu) \in O(\beta X)$  if and only if  $\mu \in O(\beta Y)$ .

The order-preserving functional  $e_X^Y(\mu)$  is said to be the *extension* of the given order-preserving functional  $\mu$ , and the map  $e_X^Y: W(\beta Y) \rightarrow W(\beta X)$  is called the *extension operator*.

**Lemma 2.** Let  $Y$  be a  $C^*$ -embedded subspace of a Tychonoff space  $X$ . Then  $r_Y^X \circ e_X^Y = id_{W(\beta Y)}$ .

**Proof.** If  $\mu \in W(\beta Y)$  then by virtue of Proposition 2  $e_X^Y(\mu) \in W(\beta X)$ . From Proposition 1 it follows that  $r_Y^X(e_X^Y(\mu)) \in W(\beta Y)$ . According to the construction of the restriction operator, the restriction  $r_Y^X(e_X^Y(\mu))$  of the order-preserving functional  $e_X^Y(\mu)$  is defined on  $C = \{\psi|_Y: \psi \in C_b(X)\} \equiv C_b(Y)$ . But according to (2) and (4) we have  $r_Y^X(e_X^Y(\mu))(\varphi) = \mu(\varphi)$  for each  $\varphi \in C_b(Y)$ . Lemma 2 is proved.  $\square$

**Lemma 3.** Let  $Y$  be a  $C^*$ -embedded subspace of a Tychonoff space  $X$ . An order-preserving functional  $\mu \in W(\beta Y)$  is  $\tau$ -smooth if and only if  $e_X^Y(\mu) \in W(\beta X)$  is a  $\tau$ -smooth order-preserving functional.

**Proof.** Let  $\mu \in W_\tau(Y)$  be an arbitrary order-preserving functional and let  $\{\varphi_\alpha\} \subset C_b(X)$  be a net such that  $\varphi_\alpha \downarrow 0_X$ . Then  $\varphi_\alpha|Y \downarrow 0_Y$ . Hence,  $e_X^Y(\mu)(\varphi_\alpha) = \mu(\varphi_\alpha|Y) \rightarrow 0$ . So,  $e_X^Y(\mu) \in W_\tau(X)$ .

Let us establish the converse statement. Let  $\mu \in W(\beta Y)$  be an arbitrary order-preserving functional such that  $e_X^Y(\mu) \in W_\tau(X)$ . Then for each net  $\{\psi_\alpha\} \subset C_b(X)$  monotone decreasing to zero on  $X$  we have  $e_X^Y(\mu)(\psi_\alpha) \rightarrow 0$ . Hence from (4) it follows that for each net  $\{\psi_\alpha\} \subset C_b(X)$  satisfying  $\psi_\alpha|Y \downarrow 0_Y$  one has  $\mu(\psi_\alpha|Y) \rightarrow 0$ .

Now take an arbitrary net  $\{\varphi_\alpha\} \subset C_b(Y)$  such that  $\varphi_\alpha \downarrow 0_Y$ . Since  $Y$  is  $C^*$ -embedded in  $X$  there exists a net  $\{\psi_\alpha\} \subset C_b(X)$  such that  $\psi_\alpha|Y = \varphi_\alpha$  for all  $\alpha$ . Hence,  $\psi_\alpha|Y \downarrow 0_Y$  and therefore  $\mu(\varphi_\alpha) = \mu(\psi_\alpha|Y) \rightarrow 0$ . Thus  $\mu \in W_\tau(Y)$ . Lemma 3 is proved.  $\square$

Note that for a compact  $X$  each order-preserving weakly additive functional  $\mu : C(X) \rightarrow \mathbb{R}$  has a (continuous) order-preserving weakly additive extension  $\mu' : B(X) \rightarrow \mathbb{R}$  with  $\mu'(1_X) = \mu(1_X)$  [5]. Here  $B(X)$  is the space of all bounded functions equipped with the uniform convergence topology. As we have noted above for each Tychonoff space  $X$  the normed spaces  $C_b(X)$  and  $C(\beta X)$  are isometrically isomorphic. Therefore any  $\tau$ -smooth order-preserving functional  $\mu : C_b(X) \cong C(\beta X) \rightarrow \mathbb{R}$  may be also extended to  $B(\beta X)$  as well. We shall use the same notation for an order-preserving functional from  $W(\beta X)$  and for its extension on  $B(\beta X)$ .

Let  $Y$  be a subspace of a Tychonoff space  $X$ . Consider the following set

$$O_Y^*(X) = \{\mu \in O_\tau(X) : \mu(\chi_K) = 0 \text{ for every compact } K \subset X \text{ such that } K \cap Y = \emptyset\},$$

where  $\chi_K$  is the characteristic function of the set  $K$ .

The equalities (2) and (4) imply as easily seen, the following

**Proposition 3.** Let  $Y$  be a  $C^*$ -embedded subspace of a Tychonoff space  $X$ . Then  $e_X^Y \circ r_Y^X|O_Y^*(X) = id_{O_Y^*(X)}$ .

Lemmas 2, 3 and Proposition 3 yield that for a  $C^*$ -embedded subspace  $Y$  of a Tychonoff space  $X$  the following equalities hold

$$e_X^Y(O_\tau(Y)) = O_Y^*(X), \quad r_Y^X(O_Y^*(X)) = O_\tau(Y).$$

These equalities imply the following

**Proposition 4.** For any Tychonoff space  $X$  the maps

$$e_{\beta X}^X : O_\tau(X) \rightarrow O_X^*(\beta X)$$

and

$$r_X^{\beta X} : O_X^*(\beta X) \rightarrow O_\tau(X)$$

are mutually inverse homeomorphisms.

The next statement is the key result of the present paper.

**Theorem 1.** For an arbitrary Tychonoff space  $X$  and for every its compactification  $bX$  the spaces  $O_X^*(\beta X)$  and  $O_X^*(bX)$  are homeomorphic.

**Proof.** First recall that a continuous map  $f : b_1X \rightarrow b_2X$  between compactifications  $b_1X$  and  $b_2X$  of the given Tychonoff space  $X$  is called *natural*, if  $f(x) = x$  for all  $x \in X$  [5, p. 47].

Let  $X$  be a Tychonoff space, and suppose that  $bX$  is its arbitrary compact extension. Let  $f : \beta X \rightarrow bX$  be a natural map. Assume that  $\mu \in O_X^*(\beta X)$  and  $O(f)(\mu) = \nu$ . Consider an arbitrary compact set  $F \subset bX \setminus X$ . By virtue of Theorem 3.5.7 in [6, p. 220] the inclusion  $f^{-1}(F) \subset \beta X \setminus X$  holds. Put  $K = f^{-1}(F)$ . Then  $f(K) = F$  and  $\chi_K = \chi_F \circ f$ . We have

$$\nu(\chi_F) = O(f)(\mu)(\chi_F) = \mu(\chi_F \circ f) = \mu(\chi_K) = 0.$$

So,  $O(f)(O_X^*(\beta X)) \subset O_X^*(bX)$ . In other words the following restriction map is correctly defined

$$O(f)|O_X^*(\beta X) : O_X^*(\beta X) \rightarrow O_X^*(bX). \quad (5)$$

The map  $O(f)|O_X^*(\beta X)$  is continuous as the restriction of the continuous map  $O(f) : O(\beta X) \rightarrow O(bX)$ .

Let  $\mu \in O(\beta X) \setminus O_X^*(\beta X)$ . Then there exists a compact set  $K \subset \beta X \setminus X$  such that  $\mu(\chi_K) \neq 0$ . Applying Theorem 3.5.7 from [6, p. 220], we obtain  $f(K) \subset bX \setminus X$  and

$$O(f)(\mu)(\chi_{f(K)}) = \mu(\chi_{f(K)} \circ f) = \mu(\chi_K) \neq 0.$$

Hence,  $O(f)(\mu) \in O(bX) \setminus O_X^*(bX)$ . So  $O(f)(O_X^*(\beta X)) = O_X^*(bX)$  since the map  $O(f)$  is surjective, i.e. the map (5) is surjective.

Note that for every compact extension  $bX$  of a Tychonoff space  $X$  the inclusion

$$O_\beta(X) \subset O(bX),$$

is true. Thus, for every Tychonoff space  $X$  and its compact extension  $bX$  one has

$$O_\beta(X) \subset O(\beta X) \cap O(bX). \quad (6)$$

From this it follows

$$O_\beta(X) \subset O_X^*(\beta X) \cap O_X^*(bX).$$

Lemma 4 from [2] and the inclusion (6) imply

$$O(f)(\mu) = \mu \quad (7)$$

for each order-preserving functional  $\mu \in O_\beta(X)$ .

Now we need the density lemma for order-preserving functionals. Recall that the *density* of a topological space  $X$  is the least cardinal number of the form  $|A|$  where  $A$  runs over everywhere dense subsets of the space  $X$ , and  $|A|$  denotes the cardinality of the set  $A$ . The density of a topological space  $X$  is denoted by  $d(X)$ .

For a Tychonoff space  $X$  put

$$O_\omega(X) = \{\mu \in O(\beta X) : \text{supp } \mu \subset X \text{ and } \text{supp } \mu \text{ is a finite set}\}.$$

The following statement may be considered as a version of the density Lemma 1.4 from [8] for order-preserving functionals.

**Lemma 4.** *For an infinite Tychonoff space  $X$  and for its subspace  $Y$  the set  $O_\omega(Y)$  is everywhere dense in  $O(\beta X)$  if and only if  $Y$  is everywhere dense in  $X$ .*

**Proof.** If  $Y$  is not everywhere dense in  $X$  then there exists a nonempty open set  $U \subset X$  such that  $U \cap Y = \emptyset$ . Take  $x \in U$ . Consider a basic neighborhood  $\langle \delta_x; \varphi; \varphi(x) \rangle$ , where  $\delta_x$  is the Dirac measure, defined by  $\delta_x(\psi) = \psi(x)$ ,  $\psi \in C_b(X)$ , and  $\varphi \in C_b(X)$  is a function such that  $\varphi(x) > 0$  and  $\varphi(y) = 0$  for all  $y \in X \setminus U$ . Then it is clear that  $\langle \delta_x; \varphi; \varphi(x) \rangle \cap O_\omega(Y) = \emptyset$ .

Let now  $Y$  be an everywhere dense in  $X$ . Then we have

$$O_\omega(Y) \subset O_\beta(Y) \subset (\text{since } O_\beta \text{ is monomorphic [3]}) \subset O_\beta(X),$$

and hence,  $O_\omega(Y) \subset O_\omega(X)$ . Let  $\mu \in O_\omega(X)$  be an arbitrary order-preserving functional, and let  $\langle \mu; \varphi_1, \dots, \varphi_k; \varepsilon \rangle$  be a neighborhood of  $\mu$ . Suppose that  $\text{supp } \mu = \{x_1, \dots, x_n\}$ . One can choose a set  $\{y_1, \dots, y_s\} \subset Y$  and an order-preserving functional  $\nu \in O_\omega(Y)$  such that the following conditions hold:

- (i)  $\text{supp } \nu = \{y_1, \dots, y_n\}$ ;
- (ii)  $|\nu(\varphi_i) - \mu(\varphi_i)| < \varepsilon$ ,  $i = 1, \dots, k$ . This implies  $\nu \in \langle \mu; \varphi_1, \dots, \varphi_k; \varepsilon \rangle$ , i.e. the set  $O_\omega(Y)$  is everywhere dense in  $O_\omega(X)$ .

From the above it follows in particular that the set  $O_\omega(X)$  is everywhere dense in  $O_\omega(\beta X)$ . On the other hand according to Proposition 3 in [2]  $O_\omega(\beta X)$  is everywhere dense in  $O(\beta X)$ . Therefore  $O_\omega(X)$  is everywhere dense in  $O(\beta X)$  and thus,  $O_\omega(Y)$  is everywhere dense in  $O(\beta X)$ . Lemma 4 is proved.  $\square$

According to Lemma 4 the set  $O_\beta(X)$  is everywhere dense in the spaces  $O(\beta X)$  and  $O(bX)$ . Hence,  $O_\beta(X)$  is everywhere dense in the sets  $O_X^*(\beta X)$  and  $O_X^*(bX)$ .

Now let us show that the map (5) is one-to-one. For this purpose take an arbitrary order-preserving functional  $\nu \in O_X^*(bX)$ . Suppose that there exist order-preserving functionals  $\mu_1, \mu_2 \in O_X^*(\beta X)$  such that  $\mu_1 \neq \mu_2$  and  $O(f)(\mu_1) = O(f)(\mu_2) = \nu$ . Let  $\{\mu_\alpha^i\} \subset O_\beta(X)$ ,  $i = 1, 2$ , be two nets converging to the functionals  $\mu_1$  and  $\mu_2$ , respectively. Since the map  $O(f)|_{O_X^*(\beta X)} : O_X^*(\beta X) \rightarrow O_X^*(bX)$  is continuous, the nets  $\{O(f)(\mu_\alpha^i)\}$ ,  $i = 1, 2$ , converge to  $\nu$ . On the other hand according to (7) one has  $O(f)(\mu_\alpha^i) = \mu_\alpha^i$  for  $i = 1, 2$  and for all  $\alpha$ . Hence,

$$\mu_1 = \lim_\alpha \mu_\alpha^1 = \nu = \lim_\alpha \mu_\alpha^2 = \mu_2.$$

We thus obtain a contradiction which shows that our assumption is false.

From the above, in particular, it follows that the map

$$(O(f)|_{O_X^*(\beta X)})^{-1} : O_X^*(bX) \rightarrow O_X^*(\beta X),$$

inverse to (5), is also continuous. Thus, the map (5) is a homeomorphism of the spaces  $O_X^*(\beta X)$  and  $O_X^*(bX)$ . Theorem 1 is then completely proved.  $\square$

Since each Tychonoff space  $X$  has a compact extension  $bX$  such that  $w(X) = w(bX)$ , according to Proposition 4 and Theorem 1 we have

$$O_\tau(X) \cong O_{\beta X}^*(X) \cong O_{bX}^*(X) \subset O(bX). \quad (8)$$

Then Proposition 0 implies the following

**Corollary 1.** *The functor  $O_\tau$  preserves the weight of every infinite Tychonoff space  $X$ , i.e.  $w(O_\tau(X)) = w(X)$ .*

According to (1) we have

**Corollary 2.** *The functor  $O_R$  preserves the weight of every infinite Tychonoff space  $X$ , i.e.  $w(O_R(X)) = w(X)$ .*

Thus,

$$w(O_\beta(X)) = w(O_R(X)) = w(O_\tau(X)) = w(X)$$

for every infinite Tychonoff space  $X$ .

Let now  $X$  be a metrizable space satisfying the second axiom of countability. Then there exists a compact extension  $bX$  such that  $w(bX) = \aleph_0$ . Since a compact is metrizable if and only if it satisfies the second axiom of countability, according to Proposition 0 the space  $O(bX)$  is a metrizable compact. Hence, according to (8) we obtain the following

**Corollary 3.** *Let  $X$  be a metrizable space satisfying the second axiom of countability. Then the space  $O_\tau(X)$  of  $\tau$ -smooth order-preserving functionals and the space  $O_R(X)$  of Radon order-preserving functionals are metrizable.*

Note that from Lemma 4 one can also obtain a strengthened version of Theorems 1.8 and 2.6 from [1].

Let  $X$  be an infinite Tychonoff space. By virtue of the inclusions (1) and  $O_\omega(X) \subset O_\beta(X)$  it follows that  $O_\beta(X)$  is everywhere dense in the spaces  $O_R(X)$ ,  $O_\tau(X)$  and  $O(\beta X)$ . On the other hand according to results of [3] one has  $d(O_\beta(X)) \leq d(X)$  for every infinite Tychonoff space  $X$ . Thus, the strengthening of Theorems 1.8 and 2.6 from [1] may be stated as follows

**Corollary 4.** *The density of an infinite Tychonoff space does not increase under the functors:*

- $O(\beta \cdot)$  of all order-preserving functionals,
- $O_\tau$  of  $\tau$ -smooth order-preserving functionals,
- $O_R$  of Radon order-preserving functionals and
- $O_\beta$  of order-preserving functionals with compact supports.

Moreover, for every infinite Tychonoff space  $X$  we have

$$d(O(\beta X)) \leq d(O_\tau(X)) \leq d(O_R(X)) \leq d(O_\beta(X)) \leq d(X).$$

Let us now recall the following notion.

**Definition 3.** (See [3].) The *weak density*  $wd(X)$  of a topological space  $X$  is the least cardinal number  $\tau$  such that  $X$  has a  $\pi$ -base which is the union of  $\tau$  centered families of open sets in  $X$ .

We need the following properties of the weak density [3]:

- (A) If  $Y$  is everywhere dense in  $X$  then  $wd(Y) = wd(X)$ ;
- (B) If  $X$  is compact then  $wd(X) = d(X)$ .

The property (A) of the weak density, Lemma 4 and the inclusions (1) imply:

**Proposition 5.** *For each Tychonoff space  $X$  one has*

$$wd(O_\omega(X)) = wd(O_\beta(X)) = wd(O_R(X)) = wd(O_\tau(X)) = wd(O(\beta X)). \quad (9)$$

Moreover according to (A) and (B) one has  $wd(X) = wd(\beta X) = d(\beta X)$  and  $wd(O(\beta X)) = d(O(\beta X))$ . Therefore Lemma 4 and Proposition 5 imply

**Corollary 5.** *The weak density of an infinite Tychonoff space does not increase under the functors  $O(\beta \cdot)$ ,  $O_\tau$ ,  $O_R$  and  $O_\beta$ .*

In connection with Proposition 5 the following question arises

**Question.** Are equalities similar to (9) valid for the density of a Tychonoff space  $X$ ?

The next result gives a positive answer for a particular case.

**Proposition 6.** *The space  $O_R(X)$  is separable if and only if the space  $O_\beta(X)$  is separable.*

**Proof.** Since  $O_\beta(X)$  is everywhere dense in  $O_R(X)$  one has the inequality  $d(O_R(X)) \leq d(O_\beta(X))$ . Let us show that the opposite inequality is also true.

Let  $\{\mu_n\} \subset O_R(X)$  be a countable everywhere dense subset of order-preserving functionals. For every order-preserving functional  $\mu_n$  and each positive integer  $m$  there exists a compact set  $K_{n,m} \subset X$  such that

$$\mu_n(\varphi) < \frac{1}{m}, \quad (10)$$

where  $\varphi \in C_b(X)$  is an arbitrary function satisfying the following inequalities

$$0 \leq \varphi \leq \chi_{(X \setminus K_{n,m})}. \quad (11)$$

Define an order-preserving functional  $\mu_{n,m}$  on  $C_b(X)$  by the formula

$$\mu_{n,m} = r_{K_{n,m}}^X(\mu_n). \quad (12)$$

Then  $\mu_{n,m} \in O(K_{n,m}) \subset O_\beta(X)$ . We have

$$\begin{aligned} |\mu_n(\varphi) - \mu_{n,m}(\varphi)| &= (\text{according to (12)}) = |\mu_n(\varphi) - r_{K_{n,m}}^X(\mu_n)(\varphi)| \\ &= |\mu_n(\varphi) - r_{K_{n,m}}^X(\mu_n)(\varphi|_{K_{n,m}})| = (\text{according to (11)}) \\ &= |\mu_n(\varphi)| < (\text{according to (10)}) < \frac{1}{m} \end{aligned}$$

for all  $n, m$ . From this it follows that the sequence  $\{\mu_{n,m}\}_{m=1}^\infty$  pointwise converges to  $\mu_n$ . Hence,  $M \equiv \{\mu_{n,m} : m, n = 1, 2, \dots\}$  is everywhere dense in  $O_R(X)$ . On the other hand,  $M \subset O_\beta(X) \subset O_R(X)$ . This means that  $M$  is everywhere dense in  $O_\beta(X)$ , i.e.  $d(O_\beta(X)) \leq d(O_R(X))$ . Proposition 6 is proved.  $\square$

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